

Complex Lagrangian mechanics with constraints

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Abstract

In this study, it is generalized the concept of Lagrangian mechanics with constraints to complex case. To be beginning , it is considered a Kählerian manifold as a velocity-phase space. Then a non-holonomic constraint is given by 1-form on it. If the form is closed, it is found that the constraint is (locally) holonomic. In the result, complex analogous of some topics in constrained Lagrangian mechanical system is concluded.

Keywords: Kählerian manifold, Constrained Lagrangian dynamics.

1 Introduction

Modern differential geometry provides a fundamental framework for studying Lagrangian mechanics. In recent years, there are many studies as some articles in [1, 2, 3] and books in [4, 5] about differential geometric methods in mechanics. It is well known that the dynamics of Lagrangian formalisms is characterized by a suitable vector field defined on the tangent bundles which are phase-spaces of velocities of a given configuration manifold. If Q is an m -dimensional

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configuration manifold and $L : TQ \rightarrow \mathbf{R}$ is a regular Lagrangian function, then there is a unique vector field ξ_L on TQ such that dynamical equations

$$i_{\xi_L} \Phi_L = dE_L, \quad (1)$$

where Φ_L is the symplectic form and E_L is the energy associated to L . The Euler-Lagrange vector field ξ_L is a semispray (or *second order differential equation*) on Q since its integral curves are the solutions of the Euler-Lagrange equations given by

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0, \quad (2)$$

where q^i and (q^i, \dot{q}^i) , $1 \leq i \leq m$, are coordinate system of Q and TQ . The triple, either (TQ, Φ_L, ξ_L) or (TQ, Φ_L, E_L) is called *Lagrangian mechanical system* on the tangent bundle TQ . Assume that (TQ, Φ_L) is symplectic manifold and $\bar{\omega} = \{\omega_1, \dots, \omega_r\}$ is a system of constraints on TQ . We call to be a *constraint* on TQ to a non-zero 1-form $\omega = \wedge^a \omega_a$ on TQ , such that \wedge^a are Lagrange multipliers. We call $(TQ, \Phi_L, E_L, \bar{\omega})$ a *regular Lagrangian system with constraints*. The constraints $\bar{\omega}$ are said to be classical constraints if the 1-forms ω_a , $1 \leq a \leq r$, are basic. Then holonomic classical constraints define foliations on the configuration manifold Q , but holonomic constraints also admit foliations on the phase space of velocities TQ . As real studies, generally a curve α satisfying the Euler Lagrange equations for Lagrangian energy E_L will not satisfy the constraints. It must be that some additional forces (or *canonical constraint forces*) act on the system in addition to the *force* dE_L for a curve α to satisfy the constraints. It is said that the quartet $(TQ, \Phi_L, E_L, \bar{\omega})$ defines a *mechanical system with constraints* if the vector field ξ given by the equations of motion

$$i_{\xi} \Phi_L = dE_L + \wedge^a \omega_a, \quad \omega_a(\xi) = 0, \quad (3)$$

is a semispray. Then, it is given Euler-Lagrange equations with constraints as follows:

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} = \wedge^a (\omega_a)_i, \quad (4)$$

The purpose of this study is to make a contribution to the modern development of Lagrangian formalisms of classical mechanics in terms of differential-geometric methods on differentiable manifolds. So, we obtain complex Euler-Lagrange equations with constraints on the Kählerian manifold. In the conclusion section, geometrical and mechanical results of constrained mechanical system have been given. The first of them is that if the distribution on TM is integrable, a system of constraints is holonomic. The second is that constrained Lagrange energy is conserved.

The present paper is structured as follows. In sections 2, it is recalled complex and Kählerian manifolds, and also Euler-Lagrange equations on Kählerian manifolds. In section 3, complex Euler-Lagrange equations with constraints on Kählerian manifold are deduced. In the final section, the geometrical and mechanical meaning of constrained complex mechanical system was given.

2 Preliminaries

In this letter, all geometric objects are assumed to be differentiable and the sum is taken over repeated indices. Now then it is assumed $1 \leq i \leq m$.

2.1 Complex manifolds

Let M be configuration manifold of real dimension m . A tensor field J on TM is called an *almost complex structure* on TM if at every point p of TM , J is endomorphism of the tangent space $T_p(TM)$ such that $J^2 = -I$. A manifold TM with fixed almost complex structure J is called *almost complex manifold*. Assume that (x_i) be coordinates of M and (x_i, y_i) be a real coordinate system on a neighborhood U of any point p of TM . Also, let us to be $\{(\frac{\partial}{\partial x^i})_p, (\frac{\partial}{\partial y^i})_p\}$ and $\{(dx^i)_p, (dy^i)_p\}$ to natural bases over \mathbf{R} of tangent space $T_p(TM)$ and cotangent space $T_p^*(TM)$ of TM , respectively.

Let TM be an almost complex manifold with fixed almost complex structure J . The manifold TM is called *complex manifold* if there exists an open covering $\{U\}$ of TM satisfying the following condition: There is a local coordinate system (x_i, y_i) on each U , such that

$$J(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i}, J(\frac{\partial}{\partial y_i}) = -\frac{\partial}{\partial x_i}. \quad (5)$$

for each point of U . Let $z_i = x_i + \mathbf{i}y_i$, $\mathbf{i} = \sqrt{-1}$, be a complex local coordinate system on a neighborhood U of any point p of TM . We define the vector fields by

$$(\frac{\partial}{\partial z^i})_p = \frac{1}{2}\{(\frac{\partial}{\partial x^i})_p - \mathbf{i}(\frac{\partial}{\partial y^i})_p\}, (\frac{\partial}{\partial \bar{z}^i})_p = \frac{1}{2}\{(\frac{\partial}{\partial x^i})_p + \mathbf{i}(\frac{\partial}{\partial y^i})_p\} \quad (6)$$

and the dual covector fields

$$(dz^i)_p = (dx^i)_p + \mathbf{i}(dy^i)_p, (d\bar{z}^i)_p = (dx^i)_p - \mathbf{i}(dy^i)_p \quad (7)$$

which represent bases of the tangent space $T_p(TM)$ and cotangent space $T_p^*(TM)$ of TM , respectively. Then the endomorphism J is shown as

$$J(\frac{\partial}{\partial z_i}) = \mathbf{i}\frac{\partial}{\partial \bar{z}_i}, J(\frac{\partial}{\partial \bar{z}_i}) = -\mathbf{i}\frac{\partial}{\partial z_i}. \quad (8)$$

The dual endomorphism J^* of the cotangent space $T_p^*(TM)$ at any point p of manifold TM satisfies $J^{*2} = -I$, and is defined by

$$J^*(dz_i) = \text{id}z_i, \quad J^*(d\bar{z}_i) = -\text{id}\bar{z}_i. \quad (9)$$

2.2 Hermitian and Kählerian manifolds

A *Hermitian metric* on an almost complex manifold with almost complex structure J is a Riemannian metric g on TM such that

$$g(JX, JY) = g(X, Y), \quad (10)$$

for any vector fields X, Y on TM . An almost complex manifold TM with a Hermitian metric is called an *almost Hermitian manifold*. If, moreover, TM is a complex manifold, then TM is called a *Hermitian manifold*.

Let further TM be a 2m-dimensional real almost Hermitian manifold with almost complex structure J and Hermitian metric g . The triple (TM, J, g) may be named an *almost Hermitian structure*. We denote by $\chi(TM)$ the set of complex vector fields on TM and by $\wedge^1(TM)$ the set of complex 1-forms on TM . Let (TM, J, g) be an almost Hermitian structure. The 2-form defined by

$$\Phi(X, Y) = g(X, JY), \quad \forall X, Y \in \chi(TM) \quad (11)$$

is called the *Kählerian form* of (TM, J, g) .

An almost Hermitian manifold is called *almost Kählerian* if its Kählerian form Φ is closed. If, moreover, TM is Hermitian, then TM is called a *Kählerian manifold*.

2.3 Complex Euler-Lagrange Equations

In this subsection, it is recalled complex Euler-Lagrange equations for classical mechanics structured on Kählerian manifold introduced in [2].

Let J be an almost complex structure on the Kählerian manifold and (z^i, \bar{z}^i) its complex coordinates. We call to be the semispray to the vector field ξ given by

$$\xi = \xi^i \frac{\partial}{\partial z^i} + \bar{\xi}^i \frac{\partial}{\partial \bar{z}^i}, \quad \xi^i = \dot{z}^i = \bar{z}^i, \quad \bar{\xi}^i = \dot{\bar{z}}^i = \bar{z}^i. \quad (12)$$

The vector field $V = J\xi$ is called *Liouville vector field* on the Kählerian manifold. We call *the kinetic energy* and *the potential energy of system* the maps given by $T, P : TM \rightarrow \mathbf{C}$ such that $T = \frac{1}{2}m_i(\dot{\bar{z}}^i)^2 = \frac{1}{2}m_i(\dot{z}^i)^2, P = m_i g h$, respectively, where m_i is mass of a mechanic system

having m particles, \mathbf{g} is the gravity acceleration and h is the origin distance of the a mechanic system on the Kählerian manifold. Then it may be said to be *Lagrangian function* the map $L : TM \rightarrow \mathbf{C}$ such that $L = T - P$ and also *the energy function* associated L the function given by $E_L = V(L) - L$.

The vertical derivation operator i_J defined by

$$i_J \omega(Z_1, Z_2, \dots, Z_r) = \sum_{i=1}^r \omega(Z_1, \dots, JZ_i, \dots, Z_r), \quad (13)$$

where $\omega \in \wedge^r TM$, $Z_i \in \chi(TM)$. The exterior differentiation d_J is defined by

$$d_J = [i_J, d] = i_J d - di_J, \quad (14)$$

where d is the usual exterior derivation.

For almost complex structure J , the closed Kählerian form is the closed 2-form given by

$$\Phi_L = -dd_J L, \quad (15)$$

such that

$$d_J : \mathcal{F}(TM) \rightarrow \wedge^1 TM$$

By means of (3), *complex Euler-Lagrange equations* on Kählerian manifold TM is found the following as:

$$\mathbf{i} \frac{\partial}{\partial t} \left(\frac{\partial L}{\partial z^i} \right) - \frac{\partial L}{\partial z^i} = 0, \quad \mathbf{i} \frac{\partial}{\partial t} \left(\frac{\partial L}{\partial \bar{z}^i} \right) + \frac{\partial L}{\partial \bar{z}^i} = 0. \quad (16)$$

3 Complex Euler-Lagrange Equations with Constraints

In this section, we shall obtain the version with constraints of complex Euler-Lagrange equations for classical mechanics structured on Kählerian manifold introduced in [2].

Let J be an almost complex structure on the Kählerian manifold and (z^i, \bar{z}^i) its complex coordinates.. Assume to be semispray to the vector field ξ given as:

$$\xi = \xi_L + \wedge^a \omega_a = \xi^i \frac{\partial}{\partial z^i} + \bar{\xi}^i \frac{\partial}{\partial \bar{z}^i} + \wedge^a \omega_a, \quad 1 \leq a \leq r, \quad (17)$$

The vector field determined by

$$V = J\xi_L = \mathbf{i}\xi^i \frac{\partial}{\partial z^i} - \mathbf{i}\bar{\xi}^i \frac{\partial}{\partial \bar{z}^i}, \quad (18)$$

is called *Liouville vector field* on the Kählerian manifold TM . The closed 2-form given by $\Phi_L = -dd_J L$ such that

$$d_J = \mathbf{i} \frac{\partial}{\partial z^i} dz^i - \mathbf{i} \frac{\partial}{\partial \bar{z}^i} d\bar{z}^i : \mathcal{F}(TM) \rightarrow \wedge^1 TM. \quad (19)$$

is found to be

$$\begin{aligned} \Phi_L = & \mathbf{i} \frac{\partial^2 L}{\partial z^j \partial z^i} dz^i \wedge dz^j + \mathbf{i} \frac{\partial^2 L}{\partial \bar{z}^j \partial z^i} dz^i \wedge d\bar{z}^j \\ & + \mathbf{i} \frac{\partial^2 L}{\partial z^j \partial \bar{z}^i} d\bar{z}^i \wedge dz^j + \mathbf{i} \frac{\partial^2 L}{\partial \bar{z}^j \partial \bar{z}^i} d\bar{z}^i \wedge d\bar{z}^j. \end{aligned} \quad (20)$$

Let ξ be the semispray given by (17) and

$$\begin{aligned} i_\xi \Phi_L = & \mathbf{i} \xi^i \frac{\partial^2 L}{\partial z^j \partial z^i} dz^j - \mathbf{i} \xi^i \frac{\partial^2 L}{\partial z^j \partial z^i} \delta_i^j dz^i + \mathbf{i} \xi^i \frac{\partial^2 L}{\partial \bar{z}^j \partial z^i} d\bar{z}^j - \mathbf{i} \bar{\xi}^i \frac{\partial^2 L}{\partial \bar{z}^j \partial z^i} \delta_i^j dz^i \\ & + \mathbf{i} \xi^i \frac{\partial^2 L}{\partial z^j \partial \bar{z}^i} \delta_i^j d\bar{z}^i - \mathbf{i} \bar{\xi}^i \frac{\partial^2 L}{\partial z^j \partial \bar{z}^i} dz^j + \mathbf{i} \bar{\xi}^i \frac{\partial^2 L}{\partial \bar{z}^j \partial \bar{z}^i} \delta_i^j d\bar{z}^i - \mathbf{i} \bar{\xi}^i \frac{\partial^2 L}{\partial \bar{z}^j \partial \bar{z}^i} d\bar{z}^j. \end{aligned} \quad (21)$$

Since the closed Kählerian form Φ_L on TM is symplectic structure, it is obtained

$$E_L = \mathbf{i} \xi^i \frac{\partial L}{\partial z^i} - \mathbf{i} \bar{\xi}^i \frac{\partial L}{\partial \bar{z}^i} - L \quad (22)$$

and hence

$$\begin{aligned} dE_L + \wedge^a \omega_a = & \mathbf{i} \xi^i \frac{\partial^2 L}{\partial z^j \partial z^i} dz^j - \mathbf{i} \bar{\xi}^i \frac{\partial^2 L}{\partial z^j \partial \bar{z}^i} dz^j - \frac{\partial L}{\partial z^j} dz^j \\ & + \mathbf{i} \xi^i \frac{\partial^2 L}{\partial \bar{z}^j \partial z^i} d\bar{z}^j - \mathbf{i} \bar{\xi}^i \frac{\partial^2 L}{\partial \bar{z}^j \partial \bar{z}^i} d\bar{z}^j - \frac{\partial L}{\partial \bar{z}^j} d\bar{z}^j + \wedge^a \omega_a. \end{aligned} \quad (23)$$

With respect to (3), if (21) and (23) is equalized, it is calculated as follows:

$$\begin{aligned} & -\mathbf{i} \xi^i \frac{\partial^2 L}{\partial z^j \partial z^i} dz^j - \mathbf{i} \bar{\xi}^i \frac{\partial^2 L}{\partial \bar{z}^j \partial z^i} dz^j + \frac{\partial L}{\partial z^j} dz^j \\ & + \mathbf{i} \xi^i \frac{\partial^2 L}{\partial z^j \partial \bar{z}^i} d\bar{z}^j + \mathbf{i} \bar{\xi}^i \frac{\partial^2 L}{\partial \bar{z}^j \partial \bar{z}^i} d\bar{z}^j + \frac{\partial L}{\partial \bar{z}^j} d\bar{z}^j = \wedge^a \omega_a \end{aligned} \quad (24)$$

Now, let the curve $\alpha : \mathbf{C} \rightarrow TM$ be integral curve of ξ , which satisfies equations

$$\begin{aligned} & -\mathbf{i} \left[\xi^j \frac{\partial^2 L}{\partial z^j \partial z^i} + \dot{\xi}^i \frac{\partial^2 L}{\partial z^j \partial z^i} \right] dz^j + \frac{\partial L}{\partial z^j} dz^j \\ & + \mathbf{i} \left[\xi^j \frac{\partial^2 L}{\partial z^j \partial \bar{z}^i} + \dot{\xi}^j \frac{\partial^2 L}{\partial z^j \partial \bar{z}^i} \right] d\bar{z}^j + \frac{\partial L}{\partial \bar{z}^j} d\bar{z}^j = \wedge^a \omega_a \end{aligned} \quad (25)$$

where $\omega_a = (\omega_a)_j dz^j + (\dot{\omega}_a)_j d\bar{z}^j$ and the dots mean derivatives with respect to the time. We infer the equations

$$\frac{\partial L}{\partial z^i} - \mathbf{i} \frac{\partial}{\partial t} \left(\frac{\partial L}{\partial z^i} \right) = \wedge^a (\omega_a)_i, \quad \frac{\partial L}{\partial \bar{z}^i} + \mathbf{i} \frac{\partial}{\partial t} \left(\frac{\partial L}{\partial \bar{z}^i} \right) = \wedge^a (\dot{\omega}_a)_i. \quad (26)$$

Thus, by *complex Euler-Lagrange equations with constraints* we may call the equations obtained in (26) on Kählerian manifold TM . Then the quartet $(TM, \Phi_L, \xi, \bar{\omega})$ is named *mechanical system with constraints*.

Conclusion

Finally, considering the above, complex analogous of the geometrical and mechanical meaning of constraints given in [5, 6] may be explained as follows.

1) Let $\overline{\omega}$ be a system of constraints on Kählerian manifold TM . Then it may be defined a distribution D on $\overline{\omega}$ as follows.

$$D(x) = \{\xi \in T_x TM \mid \omega_a(\xi) = 0, \text{ for all } a, 1 \leq a \leq r\} \quad (27)$$

Thus D is $(2m - r)$ dimensional distribution on TM . In this case, a system of complex constraints $\overline{\omega}$ is called holonomic, if the distribution D is integrable; otherwise we call $\overline{\omega}$ anholonomic. Hence, $\overline{\omega}$ is holonomic if and only if the ideal ρ of $\wedge TM$ generated by $\overline{\omega}$ is a differential ideal. Obviously (26) holds for holonomic as well as anholonomic constraints. For a system of holonomic constraints, the motion lies on a specific leaf of the foliation defined by D .

2) From (3) it is obtained equalities of

$$0 = (i_\xi \Phi)(\xi) = dE_L(\xi) = \xi(E_L), \quad (28)$$

Therefore, the Lagrangian energy E_L on Kählerian manifold TM for a solution $\alpha(t)$ of (26) is conserved.

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